PERCEPTIONS OF MATHEMATICAL UNDERSTANDING

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Australasian mathematics educators responded to a questionnaire that sought views on what mathematical understanding is, how it can be identified, and what teachers can do to develop it. This paper presents the responses to two questions about indicators of children's understanding and differences between doing and understanding. Responses were grouped as verbal, cognitive and physical indicators but it is recognised that these are inter-related.

INTRODUCTION

There is widespread rhetoric about building on children's mathematical understanding. For instance, *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990) makes the claim that because the meanings that people construct "depend upon their existing understandings, we can only take from any situation the parts that either make sense to us or can be linked to some existing ideas we have" (p. 16). This seems logical, but the statement does not suggest how teachers can find out what children's current mathematical understandings are.

For Victorian teachers, the current *Curriculum & Standards Framework* (Board of Studies, 1995) includes as one goal of mathematical activity "depth of conceptual understanding" (p. 9). Similarly, the more detailed *Mathematics Course Advice: Schools of the Future* (Directorate of School Education, 1995) claims that it "emphasises the development of conceptual understanding" (p. 9) and that the assessment activities it includes provide an opportunity for children to crystallise "essential learning in terms of understanding the essential concepts" (p. 16). It notes that planned observation is the key to teachers forming a comprehensive picture of the mathematical understanding of all students. However, neither of these documents suggests what should be observed or what the indicators of understanding might be.

Despite this emphasis in both rhetoric and practice, when I ask teachers and teacher educators what they think mathematical understanding is, the question usually draws a blank stare or flippant response. When I distributed a questionnaire to teachers avoidance was also apparent, as the questions about the nature of understanding were the ones most frequently left blank. One elusive response that I particularly enjoyed, however, was "If I understood what it is to understand, I would tell you; then you would understand what I understood. But I don't, so you don't. Do you understand?" I did!

Madison (1982) would also have understood. He has summarised the problem as one of not being able to separate the object from the subject:

> Understanding is perhaps the most difficult thing to conceptualise, and understandably because it is that which is closest to us; in the reflective act of self-understanding, subject and object, knower and known are one and the same. Understanding is in fact that which we ourselves are ... The natural error of understanding consists in its taking the object of its believing for reality. This is natural, for it is impossible for understanding to do otherwise, for it to be anything else but belief, but is a presumption ... for understanding can never prove what it believes to be reality is indeed reality itself ... But this is not our goal. We are not attempting to overstep the bounds of experience but only to interrogate experience itself. We must attempt to draw up an inventory of what we do know, on the basis of our actual, lived experience. (pp. 153—154)

The intention of this paper is to do just what Madison suggests: to draw up an inventory of what we know on the basis of our experience. It focuses on how teacher educators think that teachers can see whether children understand aspects of their mathematics.

METHODOLOGY

The results reported in this paper are from one component of a larger study entitled "Teachers' constructions of their roles in building mathematical understanding". The project is using case studies to explore teachers' ideas about mathematical understanding and the ways that it develops, as well as what the teachers do in classrooms to advance it. A survey of teachers is being used to set the case studies in a broader context.

The data presented this paper are from a pilot of the questionnaire. Draft survey questions were trialed with mathematic teacher educators, and it is on some of the results of this trial that I wish to focus. The process of data collection was very informal: the pilot was posted on the internet site with a request that anyone with time to spare complete it and/or give me feedback on its format and content. Twenty-two mathematics educators responded. Five commented on some of the questions, and seventeen completed the full questionnaire. While the final survey will seek data from teachers, the responses provided by mathematics teacher educators were of interest to me, and they have also allowed me to trial different forms of data analysis and writing up.

The survey presented three types of questions. One set asked respondents to mark on a Likert scale how well various ideas and practices match their own. A second asked them to choose responses from a given set of possibilities that best suit their ideas, and then to add any further possibilities to the set. The third type of question was more open and followed by a blank space where respondents could write points, sentences or paragraphs. These questions were asked before the matching forced choice questions; for instance, the open question on strategies teachers can use for developing mathematical understanding appeared before a section where people mark on scales their frequency of use of particular given strategies.

This paper reports results from only two open questions that relate to evidence of understanding. The respondents had already been asked to imagine a lesson where a teacher is helping a child to understand a new mathematical concept, and had described what the teacher would do to develop the child's understanding. They were now asked, *"How would the teacher know when the child understands the new concept? In other words, what do you think constitutes evidence of understanding?"*. Because many of the respondents referred to different aspects of what children "do" in this section, I have also included in this paper responses to the question, *"Do you think that there is a difference between being able to do the necessary work in a mathematics class and understanding it? If so, what is the nature of this difference?"*

The analysis aims at description, and makes no claims about significance or generalisation. Following one tradition of interpretive analysis of qualitative data, responses were sorted into what seemed to be sensible groups and subgroups. Given the small number or respondents, it is not surprising that there is little repetition, so the results are presented in full. If a response was considered to contribute equally to two categories, the relevant words were included in each group, with the symbol **...** signifying that words that have been omitted but used elsewhere.

RESULTS AND DISCUSSION

Verbal Indicators

In response to the question *"How would the teacher know when the child understands the new concept?"*, many people mentioned a child's ability to articulate ideas. This included undifferentiated talking about the concept, perhaps in response to teacher questions (see Table 1), or, more specifically, explaining it and/or teaching it to others (see Table 2).

Table 1
List of Indicators Grouped as "Ability to Talk About"

Can re-tell	Ask challenging questions to take a child out of the particular to the more general
By answers to questions	I'd be looking for things said or done to provide evidence not of complete understanding as much as understanding of components first
Monitor responses carefully	One would need as a teacher to be aware of the web (or concept map) related to the new concept—and then look for evidence of being able to ...talk about them that fit this web
Intelligent questions and conversation	When they say the right thing (I can fiddle the programming and see how they react)
Being able to articulate understanding	Answer the teacher's questions in a way that demonstrates a logical conceptual structure
Question orally, probe deeper, prompt if necessary	Being able to present their logic to other members of the group
Be able to verbalise understanding.	
Respond to probing questions with confidence	
Child asks and answers relevant questions	

Table 2
List of Indicators Grouped as "Ability to Explain"

Able to explain to teacher or another child (2)	Being able to construct a model or representation
The ability to explain clearly or teach others	Being able to explain to a peer the underlying ideas and how these fit together to make a new and useful concept
Being able to teach the concept to a peer	When they can explain it to you and others in their own words, with diagrams, etc. Defend it against alternative explanations
Can explain understanding and defend it	If they can explain how they arrived at an answer—not just the process but the reasons underlying each step
Explain the idea (2)	Explaining in a way that presents some idea of the way they are "building" their solution
If they can explain ...in their own words (2)	Being able to explain a concept clearly and some of the contexts in which is can be used
Get the student to explain the concept to me in their own language	By their ability to explain and teach. That also serves the purpose of helping to clarify and reinforce the speaker's understanding
When they can teach others (2)	Being able to explain in their own language and to a child at their own level
Explain, provide evidence	
Teach other group members or respond to their explanations sensibly	
Being able to teach the concept to another student	

It is often said that one really needs to understand something well to teach it, and that explaining often increases understanding. These beliefs were evident in some responses, and others stressed explanation being "at their own level", "in their own words", etc. However, one respondent pointed out that "Explaining is difficult for children, so sometimes we have to accept 'I just know it'; and in that case I usually test understanding with unfamiliar examples that require the application of the same or slightly extended knowledge and get them to 'think out loud'".

Cognitive Indicators

With a focus on the development of cognitive links, another group of responses to the same question related to children's abilities to make connections between concepts (see Table 3). Others used the idea of connected knowledge with reference to the ability of a child to apply knowledge across various contexts and situations (see Table 4). Few gave examples or suggested how such linkages could be observed.

Table 3
List of Indicators Grouped as "Ability to Link Concepts"

Having a network of foundational ideas	Able to use the concept in a variety of ways (not contexts but, say, being able to think of multi- plication as lots of as well as repeated addition)
See how other explanations link with their own	Being able to use a range of strategies, procedures and skills to solve a problem
Being able to develop a set of logical steps towards a desired end	Being able to articulate, explain, model, etc. as it applies to a general (rather than specific) *construct*; e.g. the properties of any pyramid, subtraction with internal zeros, etc.
Ability to link representations	

Table 4
List of Indicators Grouped as "Ability to Apply across Contexts"

Ability to apply	Being able to apply that understanding across different situations or contexts
Ability to translate to a new situation Apply/transfer the idea Be able to use it in correct contexts	Being able to apply that understanding for various relevant situations
Knowing how the maths in books is related to doing things	Being able to relationally understand a problem/situation that involves mathematical knowledge
Demonstrate over a range of problems of different types	Being able to describe or use a concept in many possible situations
Use in an unfamiliar situation	Seeing the link between the real world and the mathematical model of it
	The ability to use new knowledge with an unfamiliar problem

The literature on mathematical understanding frequently stresses the importance of the linking of concepts. Three examples follow. Lovell (1971) claimed that understanding suggests more than one-off observations or familiarity with single ideas: it suggests a broader perception and analysis of a concept, with an awareness of the connections between basic ideas, "a greater awareness of the information used, and ... a careful step-by-step process with concepts being formulated and defined" (p. 22). Skemp (1976) claimed that the process of developing relational understanding consists of "building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point" (p. 23). Piaget (1978) drew a distinction between "explicative" and "implicative" processes of comprehension (p. 213), the latter being connections between implicit concepts.

One respondent linked application to the growth of understanding, emphasising the value of "observing children using the concept—but in many cases understanding may not emerge until one goes on to use the concept later and can abstract it from the learning process". This brings to mind the work of Pirie and Kieren, who describe the "fractal" nature of understanding and the need to be able to "fold back" to previous stages in order to develop more refined concepts. Pirie and Kieren (1994) argue that in order that child's formal understandings not be disjoint, they must:

> grow from the child's potential mathematical structures and unfold from less formal image-based mathematical understandings. That is, formalising understanding is not an add-on to previous informal mathematical activity or understanding. The children ... must recognise the patterns in their informal activities which allow them to come up with and justify methods which, to an observer, are now independent of the physical actions or images upon which they were originally based. (p. 43)

Their model of mathematical understanding involve a fractal quality that is able to be observed, and they suggest that appropriate curricula follow this model so are in themselves fractal, with teachers being (a) proactive in drawing students forward to the next level of understanding; (b) invocative, by causing the students to fold back to more foundational levels; and (c) validating of the current successful process or of existing understandings.

Sierpinska (1994) claims that generalisation and synthesis interplay with identification and discrimination—four "acts" of understanding. She uses the hermeneutic notion that when discovering that prior knowledge does not easily fit with new experience (that is, we encounter as "epistemological object") then we start understanding, i.e. knowing it in a new way. This is not a straightforward process, and articulates two perspectives that are complementary but different views of the phenomenon of coming to understand:

> Not all, perhaps, but some acts of understanding are acts of overcoming epistemological obstacles. And some acts of understanding may turn out to be acts of acquiring new epistemological obstacles. A description of the acts of understanding a mathematical concept would thus contain a list of the epistemological obstacles related to that concept, providing us with fuller information about its meaning. In many cases overcoming an epistemological obstacle and understanding are just two ways of speaking about the same thing. The first is "negative" and the other "positive". Everything depends on the point of view of the observer. Epistemological obstacles look backwards, focussing attention on what was wrong, insufficient, in our ways of knowing. Understanding looks forward to new ways of knowing. (p. 28)

Physical Indicators

A final group of responses to the question involved observation of physical activity of the children—doing (see Table 5), demonstrating understanding by representation(s) (see Table 6), or performing set tasks (see Table 7).

Table 5
List of Indicators Grouped as "Ability to Do (generally)"

When students DO the right thing (and I can slightly alter what they're doing and see if they fall in a heap)	One would need to be aware of the web (or concept map) related to the new concept—and then to look for evidence of being able to do things ... that fit this web

Table 6
List of Indicators Grouped as "Ability to Represent"

Ability to represent	Being able to demonstrate or model
Being able to model	Being able to construct a model or representation
The ability to model	If they can ...demonstrate (draw, give examples)
Represent the idea	I want not only [the learner being able to *explain or describe*], but evidence of being able to *do*.

This latter group echoes Mulligan's (1995) claim that the way children represent mathematical situations is closely linked with their understanding of mathematical concepts. Mulligan wrote that "we need to examine how the teaching/learning situation can be developed to promote conceptual understanding based on children's representations" and that "this firstly involves an understanding of how children use representational thinking" (p. 290).

Similarly, Carpenter (1994) said that external representations are the evidence we have of children's internal mathematical representations and that these representations should be used to develop children's understanding further, because if conceptual understanding involves the construction of connections between representations of mathematical ideas then building connections between external representations supports the construction of more coherent and useful internal representations.

Table 7
List of Indicators Grouped as "Ability to Perform Tasks"

A well-crafted written test	Complete given problems both verbally and with written working
Correct answers	Demonstrate understanding with an unseen example
Perform a task	

The literature provides models of distinction between performance and understanding. For instance, Walkerdine (1989) contrasted "reproduction" (rote-learning and rule-following) with "production", which she claims is "real understanding" (p.274).

It was clear that most of the people who completed the survey distinguished between the ability to do and the ability to understand, and this showed up in two other sections of the questionnaire. First, in listing different types of mathematical understanding, many referred to Skemp's (1976) initial differentiation between instrumental and relational understanding. (He later added further categories.) Second, the survey asked the question *Do you think that there is a difference between being able to do the necessary work in a maths class and understanding it? If so, what is the nature of this difference?* Most respondents linked "doing" with rote processes, a distinction that is common in the literature.

I have grouped the responses as those that make a clear distinction between doing and understanding (Table 8), those that see possibilities for overlap (Table 9), and those that assume relative congruence (Table 10).

Table 8
Doing Versus Understanding

Able to do does not mean able to understand (e.g. you can do a subtraction algorithm without understanding why you do each step), and able to understand does not necessarily mean able to do (e.g. you can understand what subtraction is without being able to do difficult examples)
Doing is often drill and practice which may not lead to understanding
Doing the work can apply formulas, not understanding
Doing the work does not involve translating it to a new and previously unseen experience
Many students learn to do the prescribed questions mechanically, without understanding what they are doing or why
Much of school mathematics is doing—instrumental, rule and convention bound. You could train a monkey to do it. calculators can perform it
One is physical, the other is mental, and they are not necessarily related at all. I drive a car without understanding how it works, I understanding what square roots are but I do not know how to calculate them
They are quite different. You can do without understanding how it works, and you can understand how it works without doing

Table 9
Possible Overlap Between Doing and Understanding

Depends on activity—some require understanding, some do not.
The work (doing) is a blanket approach to engaging the mind. Aspects of this engagement might lead to or confirm understanding.
There's a difference in reality, but in theory school maths needs to aim at an ability to do with understanding at levels appropriate to the child's stage of development.

Table 10
Understanding Involves Doing

There's no such thing as understanding without doing. Observation of what a child does physically is the way we judge understanding, but there is other physical action going on that is not observable—and perhaps not even a conscious act of the child. Understanding *is* doing—but a doing that is hard to observe.

This latter position is not unlike that expressed by Pirie and Kieren (1989), who believe that "there is no such thing as understanding in the abstract", because understanding is "a process, grounded within a person, within a topic, within a particular environment" (p. 39). This contrasts with the stance of Sierpinska (1994) who identifies *acts of understanding* (e.g. explanations, validations) as being different from *understandings* (e.g. concepts, theories and problems), with the latter being the potential to experience an act of understanding when necessary as well as a prop for further development.

Voight (1994) describes mathematical understanding as a social relationship—a "theme" or network of taken-as-shared mathematical meanings, where "the teacher is ... dependant on the students' indications of understanding (and) the students are dependent on the teachers' understandings of their contributions" (p. 179). Voight suggests that this interdependence, as well as the unpredictable nature of meaning making, means that development of the theme is unpredictable. He notes that classroom routines (sometimes contradicting the teachers' intentions) function to minimise the risk of disorganisation.

CONCLUSION

One questionnaire respondent did not answer any of the questions, but wrote a statement about the folly of attempts to gather perceptions of understanding in the first place. The comment is more about the subject of the study than about particular indicators of understanding, but it makes a nice conclusion in that it summarises the difficulties of interpreting the data. It also takes us back to the object-subject dilemma outlined by Madison (1982) above. The respondent wrote:

> There's no such thing as mathematical understanding as an object—it is just a term we use to express the connections we make between some mathematical constructs that we have created—and people's stages or ways of coming to grips with those. Thus "good understanding" is used to describe the ability to demonstrate knowledge that matches commonly accepted knowledge and skills; "misunderstanding" is applied to knowledge that does not match (but might be perfectly viable for that person). "Partial understanding" is somewhere in between. But these are not different types of understanding (whether verb or noun) or even different levels of understanding. They are just *perceptions* of different levels of *agreement*. *Doing* is one way that learners can demonstrate that their understanding is similar to a teacher's (or more generally by what is valued in our society). When what is done illustrates a match, we say the person understands. When it doesn't, we say there is no, or less, understanding. But we are really talking about doing, not understanding. There is understanding there, but we can't see it—and can't know what it is. We can only observe what seems to be agreement.

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